Function Theory of a Complex Variable (E2): Exercise sheet 1 solutions

1. Note that we should assume $z \neq -1$, so the fraction is defined. In this case, we have (as usual, writing $z = x + iy$):

$$\frac{z-1}{z+1} = \frac{(z-1)(\bar{z}+1)}{|z+1|^2} = \frac{z\bar{z} - 1 + 2i\mathrm{Imz}}{|z+1|^2} = \frac{x^2+y^2-1}{(x+1)^2+y^2} + \frac{2iy}{(x+1)^2+y^2},$$

   from which we can read off the real and imaginary parts.

2. First suppose $|z| = 1$. For the fraction to be defined, we require $1 - z\bar{w} \neq 0$, i.e. $\bar{w} \neq z^{-1}$. Since $z^{-1} = \bar{z}$, this requirement is equivalent to $w \neq z$. Supposing that this is the case, we have:

$$\left|\frac{z-w}{1-z\bar{w}}\right| = \left|\frac{z-w}{z(z^{-1}-\bar{w})}\right| = \frac{|z-w|}{|z|\,|\bar{z}-\bar{w}|} = \frac{|z-w|}{|z-w|} = 1.$$

   The argument for $|w| = 1$ is similar.

3. We have that:

$$|z-w|^2 + |z+w|^2 = (z-w)(\bar{z}-\bar{w}) + (z+w)(\bar{z}+\bar{w}) = 2z\bar{z} + 2w\bar{w} = 2\left(|z|^2 + |w|^2\right).$$

4. Since the modulus is a real number, we have that

$$\left|\sum_{i=1}^{n} z_i w_i\right|^2 = \mathrm{Re}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} z_j w_j \overline{z_k w_k}\right) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} 2\mathrm{Re}(z_j w_j \overline{z_k w_k}).$$

   Now, since

$$|z-w|^2 = |z|^2 + |w|^2 - 2\mathrm{Re}z\bar{w},$$

   we have that

$$2\mathrm{Re}(z_j w_j \overline{z_k w_k}) = |z_j|^2|w_k|^2 + |z_k|^2|w_j|^2 - |z_j\overline{w_k} - z_k\overline{w_j}|^2.$$

   Hence

$$\begin{aligned}
\left|\sum_{i=1}^{n} z_i w_i\right|^2 &= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(|z_j|^2|w_k|^2 + |z_k|^2|w_j|^2 - |z_j\overline{w_k} - z_k\overline{w_j}|^2\right) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}|z_j|^2|w_k|^2 - \sum_{1\leq j<k\leq n}\left(|z_j\overline{w_k} - z_k\overline{w_j}|^2\right) \\
&= \left(\sum_{i=1}^{n}|z_i|^2\right)\left(\sum_{i=1}^{n}|w_i|^2\right) - \sum_{1\leq i<j\leq n}\left(|z_i\overline{w_j} - z_j\overline{w_i}|^2\right).
\end{aligned}$$

5. Since $i = \mathrm{cis}(\pi/2)$, these are readily checked to be equal to $\mathrm{cis}(\pi/6)$, $\mathrm{cis}(5\pi/6)$, $\mathrm{cis}(9\pi/6)$, i.e. $\pm\frac{\sqrt{3}}{2} + \frac{i}{2}$ and $-i$.

6. (a) If $z_1 = z_2$, then it is the whole of $\mathbb{C}$. Else, it is the line bisecting z_1 and z_2.
   (b) The unit circle in $\mathbb{C}$ (centred at 0).
   (c) The half-plane to the right of the line $x = c$ (writing $z = x + iy$).
   (d) Rewriting $z = x + iy$ yields the equation $y^2 = 2x + 1$, which is a parabola.

7. We have that

$$4\langle z, w\rangle = (z+\bar{z})(w+\bar{w}) - (z-\bar{z})(w-\bar{w}) = 2z\bar{w} + 2\bar{z}w = 2(z,w) + 2(w,z),$$

   which gives the first inequality. The second follows because

$$(z,w) + (w,z) = z\bar{w} + \bar{z}w = 2\mathrm{Re}z\bar{w} = 2\mathrm{Re}(z,w).$$

8. For (x,x,x) to lie on the sphere, we require $3x^2 = 1$, i.e. $x = \pm 1/\sqrt{3}$. Since it is assumed that $x \in \mathbb{R}_+$, we must have that $x = 1/\sqrt{3}$. Hence

$$|\pi(x,x,x)| = \left|\frac{x+ix}{1-x}\right| = \left|\frac{1+i}{\sqrt{3}-1}\right| = \frac{\sqrt{2}}{\sqrt{3}-1} = \frac{\sqrt{6}+\sqrt{2}}{2}.$$

Similarly,

$$|\pi(x,x,-x)| = \left|\frac{x+ix}{1+x}\right| = \left|\frac{1+i}{\sqrt{3}+1}\right| = \frac{\sqrt{2}}{\sqrt{3}+1} = \frac{\sqrt{6}-\sqrt{2}}{2}.$$

9. First, suppose $z, z' \in \mathbb{C}$. Recall that if $\pi^{-1}(z) = (x_1, x_2, x_3)$, then

$$(x_1, x_2, x_3) = \left(\frac{z+\bar{z}}{1+|z|^2}, \frac{z-\bar{z}}{i(1+|z|^2)}, \frac{|z|^2-1}{1+|z|^2}\right).$$

Hence, noting also that $x_1^2 + x_2^2 + x_3^2 = 1$,

$$\begin{aligned} d(z,z')^2 &= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3') \\ &= 2 - \frac{2\left((z+\bar{z})(z'+\bar{z}') - (z-\bar{z})(z'-\bar{z}') + (z\bar{z}-1)(z'\bar{z}'-1)\right)}{(1+|z|^2)(1+|z'|^2)}, \end{aligned}$$

and we obtain the desired expression by simplification. If $z' = \infty$, then $\pi^{-1}(z) = (0,0,1)$, and so

$$\begin{aligned} d(z,z')^2 &= 2 - 2x_3 \\ &= \frac{2(|z|^2+1) - 2(|z|^2-1)}{|z|^2+1}, \end{aligned}$$

from which we readily obtain the result. Various geometric arguments are also possible.

10. Two points on the Riemann sphere are diametrically opposite if and only if the Euclidean distance between them is equal to 2. In particular, z and z' correspond to diametrically opposite points on the Riemann sphere if and only if

$$d(z,z') = \left|\pi^{-1}(z) - \pi^{-1}(z')\right| = 2.$$

Now, if $z, z' \in \mathbb{C}$, then this requirement is equivalent to

$$\frac{|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}} = 1,$$

which can be transformed to $|z-z'|^2 = (1+|z|^2)(1+|z'|^2)$. Rewriting this as $(z-z')(\bar{z}-\bar{z}') = (1+z\bar{z})(1+z'\bar{z}')$ and expanding out yields $-z\bar{z}' - z'\bar{z} = 1 + z\bar{z}z'\bar{z}'$. This may in turn be factorised to yield $(z\bar{z}'+1)(z'\bar{z}+1) = 0$. Observe the LHS here is equal to $|z\bar{z}'+1|^2$, and hence we arrive at the equivalent expression $z\bar{z}' = -1$, as desired.
If $z = \infty$, then clearly we must have $z' \in \mathbb{C}$ (else the points do not correspond to diametrically opposite points of the sphere. Again, we must have $d(z,z') = 2$, i.e.

$$\frac{1}{\sqrt{1+|z'|^2}} = 1.$$

The only point satisfying this equation is $z' = 0$, and for this choice of z and z', we are supposing $z\bar{z}' = -1$ holds.